Math 409 Midterm 1

Name: ____________________

This exam has 3 questions, for a total of 100 points.

Please answer each question in the space provided. No aids are permitted.

Question 1. (40 pts)

In each of the following eight cases, indicate whether the given statement is true or false. No justification is necessary.

(a) Any finite subset of $\mathbb{R}$ has a least element.

Solution: True.

(b) If E is an nonempty set such that there exists a one-to-one function $f\colon \mathbb{N} \to E$, then E is countable.

Solution: False.

(c) If A is a nonempty subset of B, then there exists a surjective function $g\colon B \to A$.

Solution: True.

(d) Let A be a bounded nonempty subset of $\mathbb{R}$. If $B = \{x^3 \mid x \in A\}$, then we have $\sup B = (\sup A)^3$.

Solution: True.

(e) Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 + x$. Then $f([-1, 1]) = [0, 2]$.

Solution: False.

(f) Let E be a nonempty subset of $\mathbb{R}$. Suppose E has a finite supremum and $\sup E \notin E$. Then E is an infinite set.

Solution: True.

(g) Let A be a nonempty subset of $\mathbb{R}$. If every number in A is positive, then A has a finite infimum.

Solution: True.

(h) There does not exist a one-to-one function from $\mathbb{R}$ to $\mathbb{N}$.

Solution: True.

Question 2. (25 pts)

(a) State the Archimedean principle.

> **Solution:** If $a, b \in \mathbb{R}$ with $a > 0$, then there exists $n \in \mathbb{N}$ such that $b < na$.

(b) Prove that for a given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\frac{1}{n} < \varepsilon$$

for all $n \geq N$.

> **Solution:** Let $a = 1$ and $b = \frac{1}{\varepsilon}$, then by the Archimedean principle there exists $N \in \mathbb{N}$ such that
>
> $$b < N \cdot 1 = N.$$
>
> It follows that $b < n$ for all $n \geq N$, or equivalently
>
> $$\frac{1}{n} < \varepsilon$$
>
> for all $n \geq N$.

Question 3. (35 pts)

(a) State the completeness axiom for $\mathbb{R}$.

> **Solution:** For every nonempty subset $E \subset \mathbb{R}$, if E is bounded above, then E has a finite supremum.

(b) Let S be a bounded nonempty subset of $\mathbb{R}$, and let a and b be fixed real numbers. Define $T = \{as + b \mid s \in S\}$. Find the formulas for $\sup T$ and $\inf T$ in terms of $\sup S$ and $\inf S$. (Just the formulas, no justification is required for this part.)

> **Solution:**
>
> (1) If $a > 0$, then $\sup T = a(\sup S) + b$ and $\inf T = a(\inf S) + b$.
>
> (2) If $a = 0$, then $\sup T = \inf T = b$.
>
> (3) If $a < 0$, then $\sup T = a(\inf S) + b$ and $\inf T = a(\sup S) + b$.

(c) Let A and B be two nonempty subsets of $\mathbb{R}$. Define

$$A + B = \{a + b \mid a \in A \text{ and } b \in B\}.$$

Prove that if both A and B are bounded above, then $\sup(A+B) = \sup A + \sup B$.

> **Solution:** (1) Since $a \le \sup A$ for all $a \in A$ and $b \le \sup B$ for $b \in B$, we have
>
> $$a + b \le \sup A + \sup B$$
>
> for all $a \in A$ and $b \in B$. Thus $\sup A + \sup B$ is a upper bound of $A + B$. Thus $\sup A + \sup B \ge \sup(A + B)$.
> (2) On the other hand, by the approximation property for suprema, for $\forall \varepsilon > 0$, there exists $a \in A$ such that
>
> $$\sup A - \varepsilon/2 < a \le \sup A;$$
>
> and similarly, there exists $b \in A$ such that
>
> $$\sup B - \varepsilon/2 < b \le \sup B.$$
>
> It follows that for $\forall \varepsilon > 0$, there exists $a \in A$ and $b \in B$ such that
>
> $$\sup A + \sup B - \varepsilon < a + b \le \sup A + \sup B.$$
>
> This implies
>
> $$\sup A + \sup B - \varepsilon \le \sup(A + B)$$
>
> for all $\varepsilon > 0$, since we always have $a+b \le \sup(A+B)$. Therefore, $\sup A+\sup B \le \sup(A + B)$.
> Combining (1) and (2), we see that $\sup A + \sup B = \sup(A + B)$.